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Geophysical Sciences Laboratory Report No. 63-06

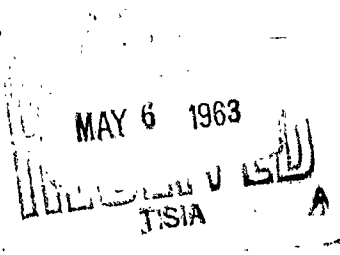
A Diabatic Weather Prediction Model with a  
Consistent Energy Integral

Joseph P. Gerrity, Jr.

Scientific Report No. 4

Sponsored by the Office of Naval Research under  
Contract Nonr 285(09)  
(Cyclone Development)

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### Abstract

Spar's [11] vertically integrated, diabatic model is modified so that it possesses an invariant mass integral of the sum of enthalpy and the kinetic energy of the non-divergent wind in the adiabatic case. The model's linear characteristics are discussed, and a series of numerical experiments utilizing idealized initial conditions is proposed.

## Introduction

The results [12] obtained with the diabatic model derived by Spar [11] indicated that diabatic processes were capable of influencing the development of a cyclone. The predictions, however, were found to be subject to a spurious anticyclogenesis. This effect was attributed to an inconsistent treatment of the two components of the curl of the vertical advection of the horizontal wind term in the vorticity equation. Arnason and Carstensen [2] had previously discussed such an effect but it was considered to be important only in hemispheric scale predictions. As a result of the apparent significance of inconsistencies of this type for even short-range predictions, we were led to consider other possible inconsistencies in the formulation of the model. Three types of inconsistency were found to exist. The first involves assumptions regarding the modeling of the vertical distribution of various model parameters. For example, the assumed vertical temperature profile is taken as invariant. This assumption is in general not consistent with the predictive equations of the model. It was concluded, however, that this kind of inconsistency is inherent in all but the simplest atmospheric models. Even a model with many information levels requires the implicit assumption of a prescribed vertical distribution within the layers separating the information levels, and this distribution (e. g. linear variation) is generally incompatible with the predictive equations. It was decided that no attempt would be made at this time to eliminate the first type of inconsistency on the assumption that it does not lead to serious errors.

Secondly, following the ideas of Lorenz [9], we found that the model equations possessed an inconsistent energy integral. (Spar [11] had previously discussed the inconsistency in the circulation integral.) It was concluded that, although such an inconsistent energy integral might not be crucial for an adiabatic model, it would deprive a diabatic model of an important balance. If spurious energy sources are present in an adiabatic version of a model, it is likely that the diabatic model will also contain spurious sinks and sources, but their nature cannot be anticipated. The major part of this paper is concerned with this problem, and presents a simplified version of the diabatic model proposed by Spar [11] which has been modified to possess a consistent energy integral. The model is derived and formulated as a set of difference equations which are to be integrated numerically using a series of idealized initial conditions. It is at this point that we come to the third type of inconsistency.

Generally, the discussion of the merits of difference schemes has been based on the order of magnitude of their truncation errors. A different but related problem is connected with the possibility that, although the differential equations of the model may possess a consistent energy integral, the finite difference equations may not preserve this property. Shuman [10] has devoted considerable effort to the study of this problem utilizing the so-called primitive equations. He found two schemes for approximating the non-linear advection terms which preserved the energy integral of his fundamental equations. However, no comparable work has been reported regarding the proper choice of difference operators for terms appearing in the



filtered equations. Gates [8] has recently reported a related difficulty encountered in the hemispheric integration of an energetically consistent two layer model with variable static stability. In an effort to minimize this error, we have incorporated in the difference equations an approximation to the Jacobian operator suggested by A. Arakawa [1].

#### 1. Some integral properties of the quasi-static equations

For an ideal, inviscid gas the hydrodynamic equations in the p-coordinate system [5] may be written

$$\frac{\partial \zeta}{\partial t} + \nabla \cdot \eta W + |k \cdot \nabla \times \left( \omega \frac{\partial W}{\partial p} \right) = 0 \quad (1.1)$$

$$\frac{\partial \delta}{\partial t} + \nabla \cdot (\eta |k \times W) + \nabla \cdot \left( \omega \frac{\partial W}{\partial p} \right) + \nabla \cdot \nabla (\phi + K) = 0 \quad (1.2)$$

$$\frac{\partial \omega}{\partial p} = -\nabla \cdot W \equiv -\delta \quad (1.3)$$

$$\frac{\partial \phi}{\partial p} = -\frac{RT}{p} \quad (1.4)$$

$$\frac{\partial T}{\partial t} + \nabla \cdot \left( T + \frac{\phi}{c_p} \right) W + \frac{\partial}{\partial p} \left( T + \frac{\phi}{c_p} \right) \omega = \nabla \cdot \frac{\phi}{c_p} \cdot W + \frac{H}{c_p} \quad (1.5)$$

Equations (1.1) and (1.2) govern the relative vorticity ( $\zeta$ ) and the divergence ( $\delta$ ) of the horizontal wind. The following notations are used:

$\eta \equiv \zeta + f$ , the absolute vorticity,

$f$ , the Coriolis parameter,

$\omega \equiv \frac{dp}{dt}$ , the vertical p-velocity,

$\nabla$ , the horizontal gradient operator,

and  $|k$ , a unit vector normal to the x-y plane directed outward from the earth.

$\phi$ ,  $p$ ,  $T$ ,  $R$ ,  $c_p$ ,  $H$ , the geopotential pressure, temperature, gas constant, specific heat at constant pressure, and heat supplied, respectively,

and  $K$ , the kinetic energy per unit mass associated with  $\mathbb{W}$ . Equation (1.3) is the equation of continuity, and equation (1.5) is the first law of thermodynamics for an ideal gas.

We propose to examine two integral properties of the set (1.1) through (1.5) for the purpose of establishing the consistency of simplifying approximations which may be introduced. The paper by Lorenz [9] serves as the guide for most of this analysis.

In what follows we shall have frequent recourse to the divergence theorem,

$$\int_A \nabla \cdot \phi \nabla \psi \, d\sigma = \oint_B \phi \frac{\partial \psi}{\partial n} \, d\ell \quad (1.6)$$

where  $d\sigma$  is an element of surface area,  $B$  is the boundary curve of the area  $A$ , and  $d\ell$  is line element of  $B$ .  $\partial\psi/\partial n$  is the derivative of  $\psi$  in the direction of the outward normal to  $B$ , and  $\psi$  and  $\phi$  are arbitrary scalars.

The following symbols will be employed.

$$\widetilde{(\quad)} \equiv \int_A (\quad) \, d\sigma \quad (1.7)$$

$$\overline{(\quad)} \equiv \int_{p_t}^{p_o} (\quad) \, dp \quad (1.8)$$

In (1.8)  $p_o$  and  $p_t$  are fixed values of the pressure,  $p$ , representing the upper and lower boundaries of the atmosphere.

If the operator,  $\widetilde{(\quad)}$ , is applied to equation (1.1) we obtain,

$$\begin{aligned}
\frac{\partial \zeta}{\partial t} + \int_B \eta \cdot \vec{n} (|\mathbf{k} \times \nabla \psi) d\ell + \int_B \eta \frac{\partial \chi}{\partial n} d\ell \\
+ \int_B \omega \frac{\partial}{\partial n} \left( \frac{\partial \chi}{\partial p} \right) d\ell - \int_B \omega \vec{n} (|\mathbf{k} \times \nabla \frac{\partial \chi}{\partial p}) d\ell = 0
\end{aligned} \tag{1.9}$$

The velocity  $\mathbb{W}$  has been represented above by the sum of a solenoidal and an irrotational vector. Thus

$$\mathbb{W} \equiv |\mathbf{k} \times \nabla \psi + \nabla \chi, \tag{1.10}$$

where  $\psi$  is a streamfunction and  $\chi$  is a velocity potential. If the horizontal wind has no component normal to  $B$ , and if  $\omega$  vanishes on  $B$ , then

$$\frac{\partial \zeta}{\partial t} \equiv 0. \tag{1.11}$$

Whenever the region  $A$  is such that the line integrals on  $B$  vanish, we shall refer to  $A$  as a "closed" region. The result (1.11) is one of the integral properties of the system of quasi-static equations.

For the formulation of the energy integral it is convenient to partition the kinetic energy of the horizontal wind into components,

$$K \equiv \frac{\mathbb{W} \cdot \mathbb{W}}{2} = K_1 + K_2 + K_3 \tag{1.13}$$

in which

$$K_1 \equiv \frac{1}{2} \nabla \psi \cdot \nabla \psi \tag{1.13}$$

$$K_2 \equiv \frac{1}{2} \nabla \chi \cdot \nabla \chi \tag{1.14}$$

and

$$K_3 \equiv \nabla \chi \cdot \nabla \psi \tag{1.15}$$

Hence

$$\frac{\partial K}{\partial t} = \nabla \psi \cdot \nabla \frac{\partial \psi}{\partial t} + \nabla \chi \cdot \nabla \frac{\partial \chi}{\partial t} + \frac{\partial}{\partial t} [\nabla \chi \cdot |\mathbf{k} \times \nabla \psi], \tag{1.16}$$

$$\frac{\partial \tilde{K}}{\partial t} = - \left[ \widetilde{\psi \frac{\partial \zeta}{\partial t}} + \widetilde{\chi \frac{\partial \phi}{\partial t}} \right] \quad (1.17)$$

The right hand side of (1.17) may be evaluated by using equations (1.1) and (1.2). Under the assumption that the line integrals vanish,

$$\widetilde{\psi \frac{\partial \zeta}{\partial t}} = \widetilde{\eta W \cdot \nabla \psi} + \widetilde{|k \cdot (\nabla \psi \times \omega \frac{\partial W}{\partial p})} \quad (1.18)$$

From (1.2),

$$\widetilde{\chi \frac{\partial \phi}{\partial t}} = \widetilde{\eta |k \times W \cdot \nabla \chi} + \widetilde{\nabla \chi \cdot \nabla K} + \widetilde{\omega \nabla \chi \cdot \frac{\partial W}{\partial p}} + \widetilde{\nabla \chi \cdot \nabla \phi} . \quad (1.19)$$

From the continuity equation,

$$\widetilde{\nabla \chi \cdot \nabla K} = \widetilde{K \frac{\partial \omega}{\partial p}} . \quad (1.20)$$

It follows that

$$\frac{\partial \tilde{K}}{\partial t} = - \left[ \widetilde{K \frac{\partial \omega}{\partial p}} + \widetilde{\omega W \cdot \frac{\partial W}{\partial p}} + \widetilde{\nabla \chi \cdot \nabla \phi} \right] . \quad (1.21)$$

From the thermodynamic energy equation (1.5),

$$\frac{\partial \tilde{c_p T}}{\partial t} = + \widetilde{\nabla \phi \cdot \nabla \chi} + \frac{\tilde{H}}{c_p} - \frac{\partial}{\partial p} (c_p T + \phi) \omega . \quad (1.22)$$

Addition of (1.22) to (1.21) leads to

$$\frac{\partial}{\partial t} \widetilde{(c_p T + K)} = - \left[ \frac{\partial}{\partial p} (K + c_p T + \phi) \omega \right] + \frac{\tilde{H}}{c_p} \quad (1.23)$$

Thus, in the adiabatic case the total energy of the system is conserved if  $\omega$  vanishes on the upper and lower boundaries.

On the assumption that the vertical velocity,  $\omega$ , vanishes at  $p_0$  and  $p_t$ , (1.21) and (1.22) yield

$$\frac{\partial \tilde{K}}{\partial t} = - \overline{\nabla \chi \cdot \nabla \phi} \quad (1.24)$$

and

$$\frac{\partial \tilde{c}_p T}{\partial t} = + \overline{\nabla \chi \cdot \nabla \phi} + \frac{\tilde{H}}{c_p} \quad (1.25)$$

$$\overline{\nabla \chi \cdot \nabla \phi} = \tilde{\omega a} \quad (1.26)$$

In view of (1.24) and (1.25), (1.26) implies that a positive conversion of potential energy into kinetic energy requires that a negative correlation exist between  $\omega$  and the specific volume. Such a correlation exists when warm air ascends and cold air sinks on the average.

The two integral theorems, derived above as equations (1.11) and equations (1.24) and (1.25) are regarded as fundamental properties of the quasi-static system. In what follows certain simplifying approximations will be introduced into the quasi-static system. We shall judge the validity of these approximations by seeing how well they preserve the integral theorems.

It is seen that if the kinetic energy of the irrotational part of the wind is deleted from the total kinetic energy in eq. (1.24), the integral theorems will hold for the general balanced quasi-static system.

Thompson [14] demonstrated the necessity and sufficiency of this approximation as a means of eliminating internal gravity oscillations in the system. However, the general balanced, quasi-static system is too complex to be attractive as a basis for an atmospheric model. In order to discuss additional simplifications, it is convenient to expand the modified version of (1.2),

$$\begin{aligned}
& -\nabla \cdot \eta \nabla \psi + \nabla \cdot \nabla \phi - \nabla \cdot \nabla K_1 \\
& + \left[ \underbrace{\nabla \cdot \eta |k \times \nabla \chi}_{(1)} + \underbrace{\nabla \cdot \omega \frac{\partial}{\partial p} |k \times \nabla \psi}_{(2)} + \underbrace{\nabla \cdot \omega \frac{\partial}{\partial p} \nabla \chi}_{(3)} + \underbrace{\nabla \cdot \nabla K_2}_{(4)} + \underbrace{\nabla \cdot \nabla K_3}_{(5)} \right] = 0
\end{aligned} \tag{1.27}$$

The bracketed terms above have usually been neglected, while the remaining three terms are considered to comprise the balanced wind equation.

We wish to determine how the neglect of the bracketed terms affects the integral theorems. For this purpose we have numbered the various terms.

Consider first the energy integral. If we premultiply each of the bracketed terms in (1.27) by the velocity potential and apply the operator,  $(\sim)$ , we obtain,

$$\left[ \underbrace{0}_{(1)} - \underbrace{\omega \nabla \chi \cdot \frac{\partial}{\partial p} |k \times \nabla \psi}_{(2)} - \underbrace{\omega \frac{\partial}{\partial p} K_2}_{(3)} - \underbrace{\nabla \chi \cdot \nabla K_2}_{(4)} - \underbrace{\nabla \chi \cdot \nabla K_3}_{(5)} \right] \tag{1.28}$$

The terms numbered (3) and (4) in equation (1.28), may be combined by use of the continuity equation. If we apply the operator,  $(\sim)$ , to the result we find that this pair vanishes. Consequently the neglect of the terms numbered (1), (3) and (4) in equation (1.27), does not require any modification of the vorticity equation in order to maintain the energy integral property of the general balanced, quasi-static system. It can be shown that the neglect of the terms numbered (2) and (5) does require that the vertical derivative appearing in the vorticity equation be modified by deleting the irrotational wind component.

Thus, if we rewrite the vorticity equation in the form,

$$\frac{\partial \zeta}{\partial t} + \nabla \cdot \eta |k \times \nabla \psi + \nabla \cdot \eta \nabla \chi + |k \cdot \nabla \times \left( \omega \frac{\partial}{\partial p} k \times \nabla \psi \right) = 0 \quad (1.29)$$

the balanced wind equation in the form

$$\nabla \cdot \eta \nabla \psi + \nabla \cdot \nabla \phi + \nabla \cdot \nabla K_1 = 0 \quad (1.30)$$

will be consistent with the energy integral

$$\frac{\partial}{\partial t} \widetilde{K}_1 = \overline{\nabla \chi \cdot \nabla \phi} \quad (1.31)$$

The modified vorticity equation (1.29), still possesses the integral property, (1.11).

A further simplification which has been widely used arises from the desire to eliminate the complexities arising from the last term in (1.29). Arnason and Carstensen [2] have shown that inconsistencies in the numerical approximation of this term can lead to deleterious results in forecasting over a large area. It is readily seen that if this term is neglected in (1.29) then for energetic consistency the term,  $\nabla \cdot \nabla K_1$ , should be eliminated in (1.30). It is probably easier to justify this modification of the system by appealing to the smallness of  $\nabla^2 K_1$  in comparison with  $\nabla^2 \phi$  within the context of equation (1.30). Nonetheless, the system,

$$\begin{aligned} \frac{\partial \zeta}{\partial t} + \nabla \cdot (\eta |k \times \nabla \psi) + \nabla \cdot (\eta \nabla \chi) &= 0 \\ \nabla \cdot (\eta \nabla \psi) &= \nabla^2 \phi \end{aligned} \quad (1.32)$$

constitutes an energetically closed system with the same integral properties as the general balanced, quasi-static system.

Because the balance equation of system (1.32) is non-linear in  $\psi$  and because the relative vorticity is frequently small compared

with the absolute vorticity, many numerical prediction models have been based on the quasi-geostrophic, balance equation,

$$f_c \nabla^2 \psi = \nabla^2 \phi \quad (1.33)$$

in which  $f_c$  is given a constant, mean value. The form of the vorticity equation which is energetically consistent with (1.32) is,

$$\frac{\partial \zeta}{\partial t} + \nabla \cdot (\eta |k \times \nabla \psi) + f_c \nabla^2 \chi = 0 \quad (1.34)$$

This set also preserves the areal mean value of the relative vorticity.

One further simplification is useful in this quasi-geostrophic system. It is not necessary for consistency with the energy or vorticity integrals, but it is probably consistent with the tangent plane approximation. This simplification involves writing the second term in (1.34) as

$$\nabla \cdot (\eta |k \times \nabla \psi) = J(\psi, \zeta) + \beta_c \frac{\partial \psi}{\partial x} \quad (1.35)$$

in which  $J(\psi, \zeta)$  is the Jacobian operator and  $\beta_c$  is a constant mean value for the gradient of the Coriolis parameter.

In the preceding analyses we have not found it necessary to introduce the first law of thermodynamics. We have implied, however, that the horizontal wind appearing in the equation was modified as a result of our modification of the divergence equation.

The set of equations that has been chosen as the basis for the prediction model is the following:



$$\frac{\partial \zeta}{\partial t} + J(\psi, \zeta) + \beta_c \frac{\partial \psi}{\partial x} + f_c \nabla^2 \chi = 0 \quad (1.36)$$

$$f_c \nabla^2 \psi = \nabla^2 \phi \quad (1.37)$$

$$\frac{\partial \omega}{\partial p} = -\nabla^2 \chi \quad (1.38)$$

$$\frac{\partial \phi}{\partial p} = -\frac{RT}{p} \quad (1.39)$$

and

$$\frac{\partial T}{\partial t} + \nabla \cdot \left( T + \frac{\phi}{c_p} \right) W + \frac{\partial}{\partial p} \left( T + \frac{\phi}{c_p} \right) \omega = \nabla \cdot \frac{\phi}{c_p} W + \frac{H}{c_p} \quad (1.40)$$

Although this set is considerably simpler than the general quasi-static equations, the fact that it possesses the integral properties discussed above recommends it as a useful system for studying certain simpler aspects of the transformation of energy and growth of circulation within the atmosphere.

## 2. Derivation of the integrated model

In the preceding section we considered several simplified versions of the hydrodynamic equations. A simple system that possesses an appropriate energy integral was adopted for use in this study. We shall now discuss a model formulation of these equations in which we follow in many essential respects the approach used by Spar [11]. This is a particular type of the thermotropic-model developed by Thompson [13]. (Another such model, which is somewhat more general than Spar's, was used by Berkofsky [4], but no numerical experiments with that model have been reported.)

The thermotropic models are based on the assumption that the isobaric temperature gradient does not change in direction but may vary in magnitude as pressure changes. One may express this symbolically as

$$\nabla T(x, y, p, t) = A(p) \nabla \bar{T}(x, y, t) \quad (2.1)$$

in which we have used the "bar-operator", defined by

$$(\bar{\phantom{x}}) \equiv \frac{1}{p_o - p_t} \int_{p_o}^{p_t} (\phantom{x}) dp \quad (2.2)$$

Since this operator will find frequent use later on, it is advantageous to digress for a moment to discuss it. If we imagine the portion of the atmosphere for which we wish to carry out our predictions to be contained between two pressure surfaces  $p_o$  and  $p_t$  ( $p_o > p_t$ ), then the mean value of any property of the model atmosphere may be defined by (2.2). In this model we shall use  $p_o = 1000$  mb and  $p_t = 200$  mb, and also assume that these pressure boundaries are rigid. The latter assumption implies that  $\omega (\equiv dp/dt)$  vanishes at  $p_o$  and  $p_t$ . The value of any parameter at  $p = p_o$  will be designated by a subscript,  $o$ . The difference between the mean value of a parameter and its value at  $p = p_o$  will be denoted by a subscript,  $T$ .

Returning to the thermotropic type model, we may observe that the variation of temperature prescribed by (2.1) is sufficient to allow a simple form of baroclinic development. By virtue of (2.1) we have upon integration, the general temperature distribution,

$$T(x, y, p, t) = A(p) \bar{T}(x, y, t) + F(x, y, t) . \quad (2.3)$$

Following Spar we set  $F(x, y, t) \equiv 0$ . The specification of  $A(p)$  may

be made in the form

$$A(p) = a(p/p_0)^b \quad (2.4)$$

in which  $a$  and  $b$  are arbitrary, but not independent. Using the NACA standard atmosphere as a guide, the parameter,  $b$ , was given the value, 0.200. The requirement that the mean value of  $A(p)$  be unity may then be used to specify  $a$ , which was found to be 1.123.

The vorticity equation is

$$\frac{\partial \zeta}{\partial t} = -J(\psi, \zeta) - \beta_c \frac{\partial \psi}{\partial x} - f_c \nabla^2 \chi \quad (2.5)$$

in which  $\zeta$  is the relative vorticity of the horizontal velocity,  $\psi$  is the streamfunction of the horizontal velocity,  $\beta_c$  ( $1.45 \times 10^{-13} \text{ cm}^{-1} \text{ sec}^{-1}$ ) is a constant value assigned to the variation of the Coriolis parameter,  $f_c$  ( $10^{-4} \text{ sec}^{-1}$ ) is a constant value assigned to the Coriolis parameter, and  $\chi$  is the velocity potential of the horizontal velocity.

The divergence equation is replaced by an equation of balance which is

$$f_c \nabla^2 \psi = \nabla^2 \phi \quad (2.6)$$

in which  $\phi$  ( $= g_0 z$ ) is the geopotential. In what follows we will use (2.6) in integrated form, viz.,

$$f_c \psi = \phi \quad (2.7)$$

The hydrostatic equation for an ideal gas may be written as

$$\frac{\partial \phi}{\partial p} = - \frac{RT}{p} = -a \quad (2.8)$$

in which  $T$  is the temperature,  $p$  the pressure,  $a$  the specific volume, and  $R$  is the gas constant for dry air for which we will use the value  $2.87 \times 10^6 \text{ ergs gm}^{-1} \text{ deg}^{-1}$ .

The continuity equation is

$$\frac{\partial \omega}{\partial p} = -\nabla^2 \chi \quad (2.9)$$

in which  $\omega$  is the vertical  $p$ -velocity ( $\frac{dp}{dt} \equiv \omega$ ).

The first law of thermodynamics is

$$\frac{\partial T}{\partial t} = -J(\psi, T) - \nabla \cdot \left( T + \frac{\phi}{c_p} \right) \nabla \chi - \frac{\partial}{\partial p} \left( T + \frac{\phi}{c_p} \right) \omega + \nabla \chi \cdot \nabla \frac{\phi}{c_p} + \frac{H}{c_p}, \quad (2.10)$$

in which  $c_p$  is the specific heat at constant pressure ( $c_p = 1.00 \times 10^7$  ergs  $\text{gm}^{-1} \text{deg}^{-1}$ ), and  $H$  is the rate at which the enthalpy is being changed by diabatic processes.

We shall also require equations to specify the value of  $H$  but these are best postponed until after the integrated model equations have been derived.

By using (2.4) and (2.3) in the hydrostatic equation, (2.8), we may arrive at the model representation of the geopotential, viz.,

$$\phi(x, y, p, t) = \bar{\phi}(x, y, t) + E(p) \phi_T(x, y, t), \quad (2.11)$$

in which

$$E(p) = \left[ \frac{1 - A(p)}{a - 1} \right] \quad (2.12)$$

From (2.7) and (2.12), it follows that

$$f_c \bar{\psi} = \bar{\phi}, \quad (2.13)$$

$$f_c \psi_T = \phi_T, \quad (2.14)$$

and

$$\psi(x, y, t, t) = \bar{\psi}(x, y, t) + E(p) \psi_T(x, y, t). \quad (2.15)$$

From (2.14), (2.11), (2.12), and (2.8), we may derive the relationship,

$$\psi_T = m \bar{T} \quad (2.16)$$

in which

$$m \equiv \frac{(a-1)R}{bf_c} = 1.76 \times 10^{10} \text{ cm}^2 \text{ sec}^{-1} \text{ deg}^{-1} . \quad (2.17)$$

The results outlined above are as far as we can go in modeling the parameters of the equations (2.5) through (2.10). We therefore make an additional assumption which here proves to be convenient, and in our analysis of the model's energetics proves to be quite essential. It consists of the assumption that the velocity potential,  $\chi$ , possesses a distribution analogous to that of the streamfunction, viz.,

$$\chi(x, y, t) = \bar{\chi}(x, y, t) + E(p)\chi_T(x, y, t) . \quad (2.18)$$

With (2.18), the continuity equation, (2.9), and the boundary conditions we may derive the relations,

$$\bar{\chi} = 0 , \quad (2.19)$$

$$\omega = F(p)\bar{\omega} , \quad (2.20)$$

$$\nabla^2 \chi_T = -\gamma \bar{\omega} , \quad (2.21)$$

in which

$$\gamma \equiv \left\{ \left[ \int_{p_0}^p E(p) dp \right] \right\}^{-1} = 6.33 \times 10^{-3} \text{ mb}^{-1} , \quad (2.22)$$

and

$$F(p) = \gamma \int_{p_0}^p E(p) dp . \quad (2.23)$$

The mean vorticity equation is derived by applying the "bar operator" to equation (2.5). The result is

$$\frac{\partial}{\partial t} \bar{\zeta} = -J(\bar{\psi}, \bar{\zeta}) - \overline{E^2} J(\psi_T, \zeta_T) - \beta_c \frac{\partial \bar{\psi}}{\partial x} , \quad (2.24)$$

in which

$$\overline{E^2} = 0.480 \quad . \quad (2.25)$$

It should be noted that this equation for the mean vorticity differs from a barotropic vorticity equation only in the appearance of the term involving the advection of shear vorticity by the shear wind. It is this term alone that can lead to development of the mean circulation.

The next model equation may be derived by applying the vorticity equation at  $p = p_0$ , expressing the value of the parameters at  $p_0$  in terms of the mean and shear parameters and then subtracting the result from the mean vorticity equation. One obtains in this manner the "shear" vorticity equation,

$$\frac{\partial \zeta_T}{\partial t} = (1 - \overline{E^2}) J(\psi_T, \zeta_T) - J(\bar{\psi}, \zeta_T) - J(\psi_T, \bar{\zeta}) - \beta_c \frac{\partial \psi_T}{\partial x} - f_c \nabla^2 \chi_T. \quad (2.26)$$

By using equation (2.21), the model continuity equation, in (2.26) we may replace it by

$$\frac{\partial \zeta_T}{\partial t} = (1 - \overline{E^2}) J(\psi_T, \zeta_T) - J(\bar{\psi}, \zeta_T) - J(\psi_T, \bar{\zeta}) - \beta_c \frac{\partial \psi_T}{\partial x} + f_c \gamma \bar{\omega}. \quad (2.27)$$

We may now consider the first law of thermodynamics, equation (2.10). Upon application of the bar-operator to (2.10), the equation reduces to the form,

$$\begin{aligned} \frac{\partial \bar{T}}{\partial t} = & - J(\bar{\psi}, \bar{T}) - \overline{EA} J(\psi_T, \bar{T}) - \overline{\Gamma(p) E(p)} \nabla^2 \chi_T \\ & + \overline{E^2} \nabla \chi_T \cdot \nabla \frac{\phi_T}{c_p} + \left( \frac{\bar{H}}{c_p} \right). \end{aligned} \quad (2.28)$$

In (2.28) we have introduced an approximation, viz., that the term

$$\left(T + \frac{\phi}{c_p}\right) \equiv \Gamma(p)$$

is a function of pressure alone. This assumption implies that the static stability is a function of pressure alone. In view of (2.16), the second Jacobian in (2.28) vanishes. We may formally introduce (2.16), (2.14), and (2.21) into (2.28) to write

$$\frac{\partial \psi_T}{\partial t} = -J(\bar{\psi}, \psi_T) + \bar{E}\bar{\Gamma} \gamma m \bar{\omega} + \frac{\bar{E}^2 f_c m}{c_p} \nabla \psi_T \cdot \nabla \chi_T + \frac{m}{c_p} \bar{H}, \quad (2.29)$$

in which the parameters have the following numerical values:

$$\bar{E}\bar{\Gamma} = 8.69 \text{ }^\circ\text{K}$$

$$\gamma = 6.33 \times 10^{-3} \text{ mb}^{-1}$$

$$m = 1.76 \times 10^{10} \text{ cm}^2 \text{ sec}^{-1} \text{ deg}^{-1}$$

$$f_c = 10^{-4} \text{ sec}^{-1}$$

$$\bar{E}^2 = 0.480$$

$$c_p = 1.00 \times 10^7 \text{ cm}^2 \text{ sec}^{-2} \text{ deg}^{-1}.$$

We note that both equations (2.29) and (2.27) involve the tendency of the shear streamfunction. By taking the Laplacian of (2.29) we may combine the result with (2.27) to derive an alternative diagnostic equation. We regard the resulting equation as governing the value of  $\bar{\omega}$  required to yield a flow which preserves both the balanced streamfunction and the diabatic temperature fields. The  $\omega$  equation may be written after some rearrangement in the form,

$$\begin{aligned} \nabla^2 \bar{\omega} - \left( \frac{f_c}{E\Gamma m} \right) \bar{\omega} = & \frac{1}{E\Gamma \gamma m} \left[ \nabla^2 (J(\bar{\psi}, \psi_T)) + (1 - \bar{E}^2) J(\psi_T, \zeta_T) - J(\bar{\psi}, \zeta_T) \right. \\ & \left. - J(\psi_T, \bar{\zeta}) - \beta_c \frac{\partial \psi_T}{\partial x} \right] - \frac{\bar{E}^2 f_c}{E\Gamma \gamma c_p} \nabla^2 (\nabla \psi_T \cdot \nabla \chi_T) - \frac{1}{E\Gamma \gamma c_p} \nabla^2 \bar{H} \quad (2.30) \end{aligned}$$

Equations (2.30), (2.26), (2.24), and (2.21) are the basic equations of the integrated model. The dependent variables are  $\bar{\psi}$ ,  $\psi_T$ ,  $\bar{\omega}$ , and  $\chi_T$ . It is necessary to close the system by giving auxiliary equations for the specification of  $\bar{H}$  in the  $\omega$ -equation. It should be noted that only two of the four equations are prognostic; the other two equations are diagnostic. Given the initial and boundary values for  $\bar{\psi}$  and  $\psi_T$ ,  $\bar{\omega}$  and  $\chi_T$  may be computed from the diagnostic equations by simultaneous iteration, subject to prescribed boundary conditions. One may then use the solutions for  $\omega$  and  $\chi_T$  to extrapolate the streamfunctions into the future by means of the prognostic equations. Before proceeding to a discussion of the method employed for computing  $H$ , we shall examine the integral properties of the equations derived above.

### 3. Integral properties of the model equations

It was shown in section 1 that the differential equations upon which the integrated model is based possess an energy integral, compatible with that for the general quasi-static equations. We shall now demonstrate that the integrated model equations also possess a suitable energy invariant. Since the parameters in the equations of the integrated model do not vary in the vertical we shall need only an integral operator,  $(\sim)$ , which we define as a surface integral in the x-y plane over a "closed" region.



The kinetic energy of the non-divergent wind,  $K$ , may be partitioned as follows

$$K = K_M + E^2 K_T + E(p) \hat{K} \quad , \quad (3.1)$$

in which

$$K_M \equiv \frac{\nabla \bar{\psi} \cdot \nabla \bar{\psi}}{2} \quad , \quad (3.2)$$

$$K_T \equiv \frac{\nabla \psi_T \cdot \nabla \psi_T}{2} \quad , \quad (3.3)$$

and

$$\hat{K} = \nabla \bar{\psi} \cdot \nabla \psi_T \quad . \quad (3.4)$$

It follows from the nature of the modeling function,  $E(p)$ , that the integrated kinetic energy,  $\bar{K}$ , is given by

$$\bar{K} = K_M + \overline{E^2} K_T \quad . \quad (3.5)$$

Upon introducing (3.2) and (3.3) into (3.5), the local time derivative of  $\bar{K}$  may be expressed as

$$\frac{\partial \bar{K}}{\partial t} = \nabla \bar{\psi} \cdot \nabla \frac{\partial \bar{\psi}}{\partial t} + \overline{E^2} \nabla \psi_T \cdot \nabla \frac{\partial \psi_T}{\partial t} \quad . \quad (3.6)$$

When the areal average of (3.6) is taken and the divergence theorem employed, we find that

$$\frac{\partial \bar{K}}{\partial t} = - \left[ \overline{\bar{\psi} \frac{\partial \bar{\psi}}{\partial t}} + \overline{E^2 \psi_T \frac{\partial \psi_T}{\partial t}} \right] \quad . \quad (3.7)$$

The right hand side of (3.7) may be evaluated by use of the vorticity equations, (2.24) and (2.27). We obtain

$$\overline{\psi \frac{\partial \bar{\zeta}}{\partial t}} = \overline{E^2} \overline{\psi_T J(\bar{\psi}, \zeta_T)} \quad , \quad (3.8)$$

and

$$\overline{\psi_T \frac{\partial \zeta_T}{\partial t}} = - \overline{\psi_T J(\bar{\psi}, \zeta_T)} + f_c \overline{\nabla \psi_T \cdot \nabla \chi_T} \quad . \quad (3.9)$$

Using (3.8) and (3.9) in (3.7) we have the result,

$$\frac{\partial \tilde{K}}{\partial t} = - \overline{E^2} f_c \overline{\nabla \psi_T \cdot \nabla \chi_T} \quad . \quad (3.10)$$

The preceding result demonstrates that the mean value of the kinetic energy of the non-divergent wind is not conserved within the model. Since the production term arises in the shear vorticity equation, we may speculate that the increase in kinetic energy appears in the shear wind first and is redistributed to the mean wind via the development term in the mean vorticity equation.

We may now consider the first law of thermodynamics governing the integrated value of the enthalpy,  $c_p \bar{T}$ . From (2.16) we may write

$$c_p \bar{T} = \frac{c_p}{m} \psi_T \quad . \quad (3.11)$$

Thus,

$$\frac{\partial}{\partial t} c_p \bar{T} = \frac{c_p}{m} \frac{\partial}{\partial t} \psi_T \quad . \quad (3.12)$$

The right side of (3.12) is readily derived from equation (2.29). We obtain the result,

$$\frac{\partial}{\partial t} c_p \bar{T} = \overline{E^2} f_c \overline{\nabla \psi_T \cdot \nabla \chi_T} + \tilde{H} \quad (3.13)$$

It should be noted in passing that the appearance of the factor  $\overline{E^2}$  in the first term on the right of (3.13) is dependent upon our having

chosen the same modeling function,  $E(p)$ , for both the non-divergent and irrotational parts of the horizontal wind. Upon combining (3.10) and (3.13), we find that

$$\frac{\partial(\bar{K} + c_p \bar{T})}{\partial t} = \bar{H} \quad (3.14)$$

which may be compared to eqs. (1.24) and (1.25).

The conservation of vorticity may also be shown to be a property of the equations of the integrated model. The mean square value of the vorticity,

$$\overline{\zeta^2} = \overline{(\bar{\zeta} + E \zeta_T)^2} = \bar{\zeta}^2 + \overline{E^2 \zeta_T^2}, \quad (3.15)$$

may be examined too. It follows from (3.15) and the vorticity equations that

$$\frac{\partial \overline{\zeta^2}}{\partial t} = 2 \overline{E^2} f_c \gamma \overline{\omega \zeta_T} \quad (3.16)$$

Since it can be readily shown that

$$\overline{E^2} \gamma = - \overline{F E'} \quad (3.17)$$

the result (3.16) compares reasonably with that for the simple hydrostatic vorticity equation (2.5), viz.,

$$\frac{\partial \overline{\zeta^2}}{\partial t} = -2 f_c \omega \overline{\frac{\partial \zeta}{\partial p}}, \quad (3.18)$$

within the framework of the integrated model. Since by (2.16) we may write

$$\zeta_T \equiv \nabla^2 \psi_T \sim \nabla^2 \bar{T} \quad (3.19)$$

Then (3.16) may be interpreted as implying that the circulation is

intensified when  $\bar{\omega}$  is correlated with the mean temperature's field of "lumpiness". For example, if cold pockets ( $\nabla^2 T > 0$ ) were on the average associated with ascending air ( $\bar{\omega} < 0$ ) or warm pockets with descending air then the circulation would be weakened. Thus in some sense we may regard the occlusion process to be contained within the model.

In connection with the approximation introduced in equation (2.28), the following analysis is offered as justification for the use of this simplification. The first law of thermodynamics for an ideal gas in quasi-static equilibrium may be written in the p-coordinate system, as

$$\frac{\partial T}{\partial t} = -J(\psi, T) - \nabla \chi \cdot \nabla T + \omega S + \frac{H}{c_p}, \quad (3.20)$$

in which

$$S \equiv - \frac{\partial}{\partial p} \left( T + \frac{\phi}{c_p} \right) \equiv - T \frac{\partial \ln \theta}{\partial p} > 0 \quad (3.21)$$

is related to the static-stability ( $\partial \theta / \partial p$ ). In most quasi-geostrophic models, two approximations are made; viz.,

$$\nabla \chi \cdot \nabla T = 0, \quad (3.22)$$

$$S = S(p) \text{ or constant.} \quad (3.23)$$

When the equation incorporating (3.22) and (3.23) into (3.20) is integrated over a closed mass of the atmosphere one finds that there can be no change in the average enthalpy for adiabatic flows.

From the general form of (3.20) the proper change in enthalpy for adiabatic flows is seen to be given by

$$\frac{\overline{\partial c_p T}}{\partial t} = -\overline{\nabla \chi \cdot \nabla c_p T} + \overline{\omega S c_p} \quad (3.24)$$

Now one of the weaknesses of an integrated model, or other models of limited vertical resolution, is an inability to specify the term  $S$  with good accuracy. If we consider the consequences of making assumption (3.23), but retaining the term  $\nabla \chi \cdot \nabla T$ , we may derive by use of the continuity equation, the result,

$$\frac{\overline{\partial c_p T}}{\partial t} = \overline{T \nabla^2 \chi} = \overline{\omega \frac{\partial T}{\partial p}} \quad (3.25)$$

The result (3.25) will reduce to

$$\frac{\overline{\partial c_p T}}{\partial t} = \overline{\omega a} \quad (3.26)$$

which is the appropriate form [see (1.26)] only if the stratification is neutral. In particular for the model temperature profile used in deriving the model parameters in section 2, (3.25) leads to

$$\frac{\overline{\partial c_p T}}{\partial t} = 0.71 \overline{\omega a} \quad (3.27)$$

which represents an underestimate of the conversion of enthalpy.

The simplification which we have used in the equation (2.28), while analogous to the procedure discussed above, leads to the proper estimate of enthalpy conversion. To elaborate on the procedure, we may first transform (3.20) into the equivalent form,

$$\begin{aligned} \frac{\partial T}{\partial t} = & -J(\psi, T) - \nabla \chi \cdot \nabla T - \frac{\partial}{\partial p} \omega \left( T + \frac{\phi}{c_p} \right) \cdot \nabla \cdot \left( T + \frac{\phi}{c_p} \right) \nabla \chi \\ & + \nabla \chi \cdot \nabla T + \nabla \chi \cdot \nabla \frac{\phi}{c_p} + \frac{H}{c_p} \end{aligned} \quad (3.28)$$

We now define

$$\Gamma \equiv \left( T + \frac{\phi}{c_p} \right) , \quad (3.29)$$

and observe that

$$S \equiv - \frac{\partial}{\partial p} \Gamma . \quad (3.30)$$

Now for adiabatic flow we obtain from (3.28) the result

$$\frac{\overline{\frac{\partial c_p T}{\partial t}}}{\overline{\frac{\partial \Gamma}{\partial t}}} = \frac{\overline{\nabla \chi \cdot \nabla \phi}}{\overline{\alpha \omega}} = \overline{\alpha \omega} . \quad (3.31)$$

The result (3.31) makes it clear that the conversion of enthalpy within a quasi-static atmosphere depends upon the work against the "pressure gradient" done by the irrotational wind.

The form of (3.28) suggests that this process may be evaluated within the framework of a semi-geostrophic model with weak vertical resolution by treating  $\Gamma$  as a function of pressure alone. It is this procedure that we adopted by our assumption in (2.28).

The choice for the numerical value of the parameter,  $\overline{\Gamma E}$ , which appears in (2.28) was derived from the mean values of the parameter  $S (\equiv - \partial \Gamma / \partial p)$  published by Gates [7] and from the model function for the vertical velocity,  $F(p)$ . To do this we made use of the relations

$$- \overline{E \Gamma} \nabla^2 \chi_T = - \overline{\Gamma \nabla^2 \chi} = \overline{\Gamma \frac{\partial \omega}{\partial p}} = \overline{\Gamma F'} \bar{\omega} , \quad (3.32)$$

$$\overline{\Gamma F'} = \overline{(\Gamma F)'} - \overline{F \frac{\partial \Gamma}{\partial p}} = \overline{F S} , \quad (3.33)$$

and the model continuity equation,

$$- \nabla^2 \chi_T = \gamma \bar{\omega} , \quad (3.34)$$

which led to

$$\overline{E\Gamma} = + \frac{1}{\gamma} \overline{FS} > 0 \quad . \quad (3.35)$$

The average winter and summer values of  $\overline{FS}$  were computed to be

$$5.53 \times 10^{-2} \text{ } ^\circ\text{K mb}^{-1}$$

and

$$4.45 \times 10^{-2} \text{ } ^\circ\text{K mb}^{-1} \text{ ,}$$

respectively. We chose to use a value biased toward winter values, viz.,  $\overline{FS} = 5.50 \times 10^{-2} \text{ } ^\circ\text{K mb}^{-1}$ . Using this value in (3.35) we obtained the model value ( $\gamma \equiv 6.33 \times 10^{-3} \text{ mb}^{-1}$ ),

$$\overline{\Gamma E} = 8.69 \text{ } ^\circ\text{K} .$$

#### 4. The diabatic term

Since the differential equations governing the integrated model possess an energy invariant of a general type, it is reasonable to employ the model in a study of the energy transformations associated with diabatic processes. In particular, we have formulated approximations for the release of latent heat by condensation and the transfer of sensible heat from the sea to the air. Because of the simple vertical structure of the model, we will be forced to confine ourselves to the influence of horizontal variation in heating upon the development of circulation.

We have made use of the relationship given by Spar [11] for the estimation of the magnitude of the sensible heat flux at the air-sea interface. In modified units we may write Spar's formula as

$$Q_s = 6.01 \times 10^{-9} V_o (T_s - T_o) \quad , \quad (4.1)$$

in which  $Q_s$  is the rate of temperature change in degrees per second,  $V_o$  is the surface wind speed in  $\text{cm sec}^{-1}$ ,  $T_s$  is the sea surface temperature, and  $T_o$  is the temperature of the surface air. To account for the reduction in wind speed in passing from 1000 mb to the surface we replace  $V_o$  in (4.1) by 80% of the wind speed at 1000 mb,  $0.8 |W_o|$ . The formula then becomes,

$$Q_s = 4.81 \times 10^{-9} |W_o| (T_s - T_o). \quad (4.2)$$

Since this formula was derived for cases in which the heat flux was directed from sea to air, we arbitrarily set  $Q_s \equiv 0$  when  $T_s < T_o$ . No heat flux is computed over the land.

In order to establish a formula for the rate of heating due to condensation, we must consider an equation for the conveyance of water vapor. The parameter used to represent the water vapor is the specific humidity,  $q$ . A continuity equation for  $q$  may be written neglecting evaporation as follows,

$$\frac{\partial q}{\partial t} = -J(\psi, q) - \nabla \cdot q \nabla \chi - \frac{\partial}{\partial p} q \omega - \left( \frac{\partial q}{\partial t} \right)_c, \quad (4.3)$$

in which  $(\partial q / \partial t)_c$  is the local rate of decrease in specific humidity due to condensation. Application of the bar operator,  $(\bar{\quad})$ , to (4.3) yields,

$$\frac{\partial \bar{q}}{\partial t} = -\overline{J(\psi, q)} - \overline{\nabla \cdot q \nabla \chi} - \left( \overline{\frac{\partial q}{\partial t}} \right)_c \quad (4.4)$$

If we assume that the distribution of specific humidity may be written as,

$$q(x, y, t, p) = L(p) \bar{q}(x, y, t), \quad (4.5)$$

then the equation becomes,



$$\frac{\partial \bar{q}}{\partial t} = -J(\bar{\psi}, \bar{q}) - \overline{LE} J(\psi_T, \bar{q}) - \overline{LE} \nabla \cdot \bar{q} \nabla \chi_T - \left( \frac{\partial \bar{q}}{\partial t} \right)_c \quad (4.6)$$

In qualitative agreement with the results of Benton and Estoque [3], a reasonable average value for the parameter,  $\overline{LE}$ , has been found to be -0.6. Because the integrated model is really incapable of accurately describing the vertical distribution of specific humidity, it was decided to simplify the term,  $\nabla \cdot \bar{q} \nabla \chi$ , by replacing  $\bar{q}$  with an average value,  $\bar{q}_c$ . By introducing the model continuity equation (2.21), and the approximation just discussed, we may re-write (4.6) as,

$$\frac{\partial \bar{q}}{\partial t} = -J(\bar{\psi}, \bar{q}) - \overline{LE} J(\psi_T, \bar{q}) + \overline{LE} \bar{q}_c \gamma \bar{\omega} - \left( \frac{\partial \bar{q}}{\partial t} \right)_c \quad (4.7)$$

Recalling that  $\overline{LE} = -0.6$ , we may note that the term involving  $\bar{\omega}$  will act as a source of  $\bar{q}$  when  $\bar{\omega} < 0$  (ascending air) and as a sink of  $\bar{q}$  when  $\bar{\omega} > 0$  (descending air). Now if the air column is locally saturated, one may expect that the Jacobian terms will rarely tend to increase  $\bar{q}$  above its local saturation value. However, if  $\bar{\omega}$  is negative, the term involving  $\bar{\omega}$  will act to increase  $\bar{q}$  above its saturation value. We are therefore led to assume that,

$$\left( \frac{\partial \bar{q}}{\partial t} \right)_c = \overline{LE} \bar{q}_c \gamma \bar{\omega}, \quad (4.8)$$

if  $\bar{\omega} < 0$  and the air is saturated.

To formulate this hypothesis quasi-analytically we introduce the parameter,  $\delta$ , defined as

$$\begin{aligned} \delta &= 0 \quad \text{if} \quad \bar{\omega} \geq 0 \quad \text{or} \quad \bar{q} < \bar{q}_s \\ \delta &= 1 \quad \text{if} \quad \bar{\omega} < 0 \quad \text{and} \quad \bar{q} \geq \bar{q}_s \end{aligned} \quad (4.9)$$

Using (4.8) and (4.9) in (4.7), we may write

$$\frac{\partial \bar{q}}{\partial t} = -J(\bar{\psi}, \bar{q}) - \overline{LE} J(\bar{\psi}_T, \bar{q}) + (1 - \delta) \overline{LE} \bar{q}_c \gamma \bar{\omega} \quad (4.10)$$

Now if  $\delta = 1$ , the rate of change in mean temperature in the column may be obtained by multiplying (4.8) by the ratio of the latent heat of vaporization,  $L_v$ , to the specific heat at constant pressure,  $c_p$ . We obtain

$$Q_c = \frac{L_v \overline{LE} \bar{q}_c \gamma \bar{\omega}}{c_p} \quad (4.11)$$

Using the values,

$$L_v = 600 \text{ cal gm}^{-1}$$

$$c_p = 0.239 \text{ cal gm}^{-1} \text{ deg}^{-1}$$

$$\overline{LE} = -0.6$$

$$\bar{q}_c = 2.5 \times 10^{-3}$$

$$\gamma = 6.33 \times 10^{-3} \text{ mb}^{-1}$$

we may write

$$Q_c = \pi |\bar{\omega}| \quad , \quad (4.12)$$

in which

$$\pi = 23.83 \times 10^{-3} \text{ deg mb}^{-1} .$$

The precipitation rate,  $P$ , may also be computed from (4.8) through the relation

$$P = \frac{(P_o - P_t)}{g \rho_l} \left( \frac{\partial \bar{q}}{\partial t} \right)_c \quad , \quad (4.13)$$

in which  $g$  is the acceleration of gravity, and  $\rho_l$  is the density of liquid water. Using (4.8) we obtain

$$P = r |\bar{\omega}| \quad (4.14)$$

in which  $r = 7.60 \times 10^{-3} \text{ cm mb}^{-1}$ .

In order to compute the saturation specific humidity from the model parameters, we adopt the approximation by Spar,

$$\begin{aligned} \bar{q}_s = 7.0 \times 10^{-4} [0.07016 + 0.07899 X - 0.01358 X^2 + 0.01757 X^3 \\ - 0.00204 X^4 + 0.00025 X^5] , \end{aligned} \quad (4.15)$$

in which

$$X = [3.35 \times 10^{-9} \psi_T - 1.26 \times 10^4] \div 400 . \quad (4.16)$$

The expressions for  $Q_s$  and  $Q_c$  derived above are incorporated into the  $\omega$ -equation (2.30) in place of the ratio,  $\bar{H}/c_p$ .

### 5. Method of numerical integration

It has been a general practice to test weather prediction models by performing numerical integrations with observed initial data and comparing the predictions with the observed fields. Although this procedure is essential when it is proposed to employ the model in routine forecasting, it has several disadvantages, one of them being the necessity for a major data collection and analysis program. Additionally, one cannot control the elements entering into the initial data, and consequently pertinent factors relating to model behavior cannot be simply isolated.

For these reasons, the model derived above is to be applied to a hypothetical atmosphere contained between fixed vertical walls parallel to the zonal direction and extending to infinity. The integration region was truncated in the zonal directions by assuming that

zonal periodicity could be employed as a boundary condition across imaginary meridionally oriented walls. Based on the limitations of computer core storage and the linearized behavior of the planetary scale waves (see section on linear analysis), the zonal dimension of the integration region was fixed at 10,000 km. The zonal walls were separated by 4500 km, which is approximately 40 degrees of latitude.

Within this channel the initial field of streamfunction is specified analytically. From the analytic functions the values of streamfunction were computed at the grid points formed by two sets of lines, parallel and orthogonal to the zonal walls respectively. The lines were spaced 250 km apart and yielded an array of  $19 \times 40$  grid points. To this array we added two columns and two rows enclosing the integration region for use in specifying the boundary conditions.

The differential equations of the model fall into two categories, two dimensional boundary value equations and tendency equations. The most difficult problem encountered is the simultaneous solution of the omega and continuity equations. Individually these equations are the familiar Helmholtz and Poisson equations respectively. For the omega equation we used Dirichlet conditions ( $\bar{\omega} = 0$ ) on the zonal boundaries and periodicity on the meridional boundaries. The periodicity condition was duplicated for the continuity equation, but the zonal boundary condition was of the Neuman type ( $\partial x / \partial y = 0$ ) expressing the vanishing of the normal component of the velocity potential there.

The vorticity equations are of the Poisson type in the stream-function tendency and are solved for boundary conditions identical with those for omega.

The tendency equations were of two types. The streamfunction tendency was computed at each grid point by the solution of the vorticity equation. The temporal extrapolation is accomplished by first forward and then centered time steps over an interval  $\Delta t = 1$  hour. The water vapor conveyance equation required the computation of the Jacobians by means of upwind difference schemes to reduce the truncation error. The tendencies were extrapolated by forward differences only.

The finite difference equations used in the numerical solution are listed below. The notations are used relative to this stencil of grid points.

$$\begin{array}{ccc} +_7 & +_2 & +_6 \\ +_3 & +_0 & +_1 \\ +_8 & +_4 & +_5 \end{array}$$

$$\nabla^2 P \equiv P_1 + P_2 + P_3 + P_4 - 4P_0 \quad (5.1)$$

$$\Delta_x P \equiv P_1 - P_3 \quad (5.2)$$

$$J_1(P, Q) \equiv (P_1 - P_3)(Q_2 - Q_4) - (P_2 - P_4)(Q_1 - Q_3) \quad (5.3)$$

$$J_2(P, Q) \equiv (P_5 - P_6)Q_1 - (P_8 - P_7)Q_3 + (P_6 - P_7)Q_2 - (P_5 - P_8)Q_4 \quad (5.4)$$

$$J_3(P, Q) \equiv (P_1 - P_2)Q_6 - (P_4 - P_3)Q_8 + (P_2 - P_3)Q_7 - (P_1 - P_4)Q_5 \quad (5.5)$$

$$J(P, Q) = \frac{1}{3} [J_1(P, Q) + J_2(P, Q) + J_3(P, Q)] \quad (5.6)$$

$$\Delta_y Q = Q_2 - Q_4 \quad (5.7)$$

$$\nabla P \cdot \nabla Q = [\Delta_x P \cdot \Delta_x Q] + [\Delta_y P \cdot \Delta_y Q] \quad (5.8)$$

In these equations, an unsubscripted variable is to be understood to apply at the point O in the stencil. The grid spacing will be denoted by the letter  $h$  ( $= 250$  km).

The mean vorticity equation:

$$\nabla^2 \frac{\partial \bar{\psi}}{\partial t} = -\frac{1}{4} J(\bar{\psi}, \bar{\xi}) - \frac{\bar{E}^2}{4} J(\psi_T, \xi_T) - \frac{\beta_c h}{2} \Delta_x \bar{\psi} \quad (5.9)$$

The shear vorticity equation:

$$\nabla^2 \frac{\partial \psi_T}{\partial t} = \frac{(1 - \bar{E}^2)}{4} J(\psi_T, \xi_T) - \frac{1}{4} J(\bar{\psi}, \xi_T) - \frac{1}{4} J(\psi_T, \bar{\xi}) - \frac{\beta_c h}{2} \Delta_x \psi_T + f_c h^2 \gamma \bar{\omega} \quad (5.10)$$

The continuity equation:

$$\nabla^2 \chi = -\gamma h^2 \bar{\omega} \quad (5.11)$$

The omega equation:

$$\begin{aligned} \nabla^2 \bar{\omega} - \left( \frac{f_c h^2}{E \Gamma m} \right) \bar{\omega} = & \frac{1}{E \Gamma \gamma m} \left[ \nabla^2 \left( \frac{1}{4h^2} J(\bar{\psi}, \psi_T) \right) + \frac{(1 - \bar{E}^2)}{4} J(\psi_T, \xi_T) - \frac{1}{4} J(\psi_T, \xi_T) \right. \\ & \left. - \frac{1}{4} J(\psi_T, \bar{\xi}) - \frac{\beta_c h}{2} \Delta_x \psi_T \right] - \frac{\bar{E}^2 f_c}{4 E \Gamma \gamma c_p h^2} \nabla^2 (\nabla \psi_T \cdot \nabla \chi_T) - \frac{1}{E \Gamma \gamma} \nabla^2 \frac{\bar{H}}{c_p} \end{aligned} \quad (5.12)$$

In the water vapor conveyance equation, we make use of upwind approximations for the Jacobian operator [6]. This will be denoted by the symbol,  $\check{J}(\psi, P)$ .

Since  $\frac{\partial \psi}{\partial x} \equiv +v$ , the northward component of the wind is  $+v$ ,

and  $\frac{\partial \psi}{\partial y} \equiv -u$ , the eastward component of the wind is  $+u$ , we first determine the sign of  $u$  and  $v$  by using a centered difference approximation for  $\partial \psi / \partial x$  and  $\partial \psi / \partial y$ . We evaluate  $\partial p / \partial x$  and  $\partial p / \partial y$  by one-sided differences in the direction from which the wind is blowing.

The water vapor conveyance equation is

$$\frac{\partial \bar{q}}{\partial t} = -\frac{1}{2h} \check{J}(\bar{\psi}, \bar{q}) - \frac{LE}{2h^2} \check{J}(\psi_T, \bar{q}) + (1 - \delta) \overline{LE} \bar{q}_c \gamma \bar{\omega} \quad (5.13)$$

The auxiliary equations are

$$\bar{\epsilon} = \frac{1}{h^2} \nabla^2 \bar{\psi} \quad (5.14)$$

$$\epsilon_T = \frac{1}{h^2} \nabla^2 \psi_T \quad (5.15)$$

$$\frac{\bar{H}}{c_p} = Q_s + Q_c \quad (5.16)$$

$$Q_s = 4.81 \times 10^{-9} \left[ \frac{1}{4h^2} \nabla(\bar{\psi} - \psi_T) \cdot \nabla(\bar{\psi} - \psi_T) \right]^{1/2} \cdot (T_s - \frac{1.123}{m} \psi_T) \quad (5.17)$$

$$Q_c = \delta \pi |\bar{\omega}| \quad (5.18)$$

The program is written to produce maps of the predicted fields at 6 hour increments, as well as a record of the energy transformations which are printed out at hourly time steps.

Because of the inherent truncation error in numerical computation, we cannot assure preservation of the integral properties of the model differential equations by their numerical analogues. We have introduced the approximation to the Jacobian operator defined in (5.6) in an attempt to preserve the integral property of the Jacobian operator.

## 6. Linear analysis of the model equations

The basic model equations, (2.21, 2.24, 2.27, and 2.29), may be subjected to a linear analysis by means of the superposition of a perturbation,

$$\begin{aligned}
\bar{\Psi} &= \hat{\Psi} \\
\psi_T &= \psi' e^{ik(x-ct)} \\
\bar{\omega} &= i\hat{\omega} \\
\chi_T &= i\chi' \\
\bar{H} &= iH'
\end{aligned} \tag{6.1}$$

upon the basic state,

$$\begin{aligned}
\bar{\Psi} &= -Uy \\
\psi_T &= -U_T y \\
\bar{\omega} &= 0 \\
\chi_T &= 0 \\
\bar{H} &= 0
\end{aligned} \tag{6.2}$$

The differential equations reduce, upon substitution of (6.1) and (6.2), to this set of algebraic equations in the amplitudes,

$$\left(C - U + \frac{\beta_c}{K^2}\right) \hat{\Psi} - \overline{E^2} U_T \psi' = 0 \tag{6.3}$$

$$\left(C - U + (1 - \overline{E^2}) U_T + \frac{\beta_c}{K^2}\right) \psi' - U_T \hat{\Psi} - \frac{f_c}{K} \chi' = 0 \tag{6.4}$$

$$(C - U) \psi' + U_T \hat{\Psi} + \frac{\overline{E\Gamma} \gamma m}{K} \hat{\omega} = + \frac{m}{c_p K} H' \tag{6.5}$$

$$\chi' - \frac{\gamma}{K^2} \hat{\omega} = 0 \tag{6.6}$$

It is to be noted that the term,  $\nabla \psi_T \cdot \nabla \chi_T$ , in equation (2.29) is of higher order in the perturbation quantities and, consequently, does not appear in (6.5).

For the present we will examine the adiabatic equations, taking



**H' to be zero.** In order to simplify the computation of the stability characteristics of this system we may first use equations (6.5) and (6.6) to eliminate  $\hat{\omega}$  and  $\chi'$  from equations (6.3) and (6.4). From (6.5) and (6.6), we may write,

$$-\frac{f_c}{K} \chi' = \frac{f_c}{E\Gamma m K^2} [(C - U)\psi' + U_T \hat{\psi}]. \quad (6.7)$$

If we set the parameter, which multiplies the bracketed terms in (6.7), equal to the dummy variable,  $\rho'$ ,

$$\rho' \equiv \frac{\rho}{K^2} \equiv \left[ \frac{f_c}{E\Gamma m K^2} \right], \quad (6.8)$$

the equations (6.3) and (6.4) may be rewritten as

$$\left( C - U + \frac{\beta_c}{K^2} \right) \hat{\psi} - \overline{E}^2 U_T \psi' = 0, \quad (6.9)$$

and

$$(\rho' - 1)U_T \hat{\psi} + \left[ (1 + \rho')(C - U) + (1 - \overline{E}^2)U_T + \frac{\beta_c}{K^2} \right] \psi' = 0 \quad (6.10)$$

The determinant of the coefficients of  $\hat{\psi}$  and  $\psi'$  is

$$\Delta = (1 + \rho')X^2 + \left[ (1 - \overline{E}^2)U_T - \rho' \frac{\beta_c}{K^2} \right] X + (\rho' - 1)\overline{E}^2 U_T^2, \quad (6.11)$$

in which

$$X \equiv \left( C - U + \frac{\beta_c}{K^2} \right). \quad (6.12)$$

Since the system, (6.9) and (6.10), is homogeneous, a non-trivial solution exists only if  $\Delta$  vanishes identically. Letting (6.11) equal zero and solving the resulting quadratic for the parameter,  $C$ , we obtain,

$$C = \left[ U - \frac{\beta_c}{K^2} \right] + \left[ \frac{\rho\beta_c}{K^4} + (\overline{E}^2 - 1)U_T \pm D^{1/2} \right] \left[ \frac{2(K^2 + \rho)}{K^2} \right]^{-1}, \quad (6.13)$$

in which the discriminant,  $D$ , is given by

$$D \equiv \left[ \frac{\rho\beta_c}{K^4} + (\overline{E}^2 - 1)U_T \right]^2 - 4 \left( \frac{\rho^2}{K^4} - 1 \right) \overline{E}^2 U_T^2. \quad (6.14)$$

If  $D < 0$ , the "phase velocity",  $C$ , will be complex and the perturbations will be composed of two linearly superimposed parts: one growing exponentially, the other decaying exponentially. The rate of growth of the amplifying part of the perturbations may be computed for various values of the model parameters, as a function of  $U_T$  and the wave length,  $2\pi/K$ .

Before discussing the graph showing the growth rate for the adiabatic case, we may briefly consider the diabatic term,  $H'$ . In the formulation of the diabatic terms given in section 4, we arrived at the relation

$$\frac{\overline{H}}{c_p} = \pi |\overline{\omega}| + \overline{Q}_s, \quad (6.15)$$

in which the first term is the measure of the release of latent heat and the second term the rate of addition of sensible heat. The sensible heat term is not susceptible to linear analysis since it is a time dependent non-linear function of the dependent variables. However, the potential influence of the latent heat release may be estimated, since it is linearly related to  $\overline{\omega}$  when condensation is occurring. The principal influence which we can examine involves the effective increase in the stability parameter,  $\rho$ . Thus  $\rho$  varies from

$$\rho = 0.64 \times 10^{-15} \text{ cm}^{-2}$$

for the no-heating case to,

$$\rho = 1.11 \times 10^{-15} \text{ cm}^{-2}$$

when condensation is occurring. By examination of equation (6.14), we may determine the short-wave cut-off to the unstable waves as a function of  $\rho$ . It is clear that  $D$  will be positive for any value of  $U_T$  provided that

$$\frac{\rho^2}{K^4} < 1 \quad . \quad (6.16)$$

Using the values of  $\rho$  given above we find that for the no-heating case all waves with wave length less than  $2.49 \times 10^3$  km are stable. With heating, the reduction in the value of  $\rho$  carries the potentially unstable region down to waves with wave length greater than  $1.85 \times 10^3$  km.

If we assume that  $D$  is negative then we may write,

$$C = C_R \pm i C_C \quad . \quad (6.17)$$

Using (6.13), it follows that

$$C_R = U - \frac{\beta_c}{K^2} + \frac{K^2}{2(K^2 + \rho)} \cdot \left( \frac{\rho\beta_c}{K^4} + (\overline{E}^2 - 1) U_T \right), \quad (6.18)$$

and

$$C_C = \frac{K^2}{2(K^2 + \rho)} \cdot \left[ 4\overline{E}^2 U_T^2 \left( \frac{\rho^2}{K^4} - 1 \right) - \left( \frac{\rho\beta_c}{K^4} + (\overline{E}^2 - 1) U_T \right)^2 \right]^{1/2} . \quad (6.19)$$

If we set  $K C_C$  equal to a constant, say  $N$ , then we may solve (6.19) for  $U_T$  as a function of  $K$ . The parameter  $N$  is the time required for wave number,  $K$ , to grow to  $l$ -times its initial amplitude for the value of  $U_T$  as a function of  $K$  by,

$$U_T = \frac{B}{A} \left[ 1 \pm \left( 1 - \frac{AC}{B} \right)^{1/2} \right], \quad (6.20)$$

in which

$$A = \left[ 2.25 - 2.0 \frac{\rho^2}{K^4} \right], \quad (6.21)$$

$$B \equiv \frac{\rho \beta_c}{2K^4} \quad (6.22)$$

$$C = \frac{\rho^2 \beta_c^2}{K^8} + \frac{4N^2 \left( 1 + \frac{\rho}{K^2} \right)^2}{K^2} \quad (6.23)$$

and  
Equation (6.20) was solved for  $U_T$ , with  $N = \frac{1}{2}$  day, 1 day, 2 days, and 10 days, and for the two values of  $\rho$  given above.  $\beta_c$  was assigned the value,  $1.45 \times 10^{-13} \text{ cm}^{-1} \text{ sec}^{-1}$ . The results are given in figure 1.

In addition to these computations we also varied the value of the parameter,  $\overline{E^2}$ . We made computations for  $\overline{E^2} = 0.333, 0.50, 0.65, 1.00, 1.25$ . As  $\overline{E^2}$  increased the wave length of maximum instability shifted to larger values and the longer waves in general became more unstable for definite values of  $U_T$ .

The stability characteristics of the linearized model are used as a guide in specifying the initial distribution of  $\overline{\psi}$  and  $\psi_T$  in the non-linear numerical integrations.

Two results obtained by following the analysis procedure employed by Wiin-Nielsen [15] may be noted. By solving the initial value problem posed by the linear equations for a simple initial distribution of the perturbations, we find that the unstable waves possess a limiting phase relationship between the mean and shear streamfunctions. If

$\theta$  represents the phase lag (positive when  $\psi_T$  lags  $\bar{\psi}$ ), the limiting value is given by

$$\theta = \arctan \left[ \frac{4\bar{E}^2 U_T^2 \left( \frac{\rho^2}{K^4} - 1 \right)}{\left[ \frac{\rho \beta_c}{K^4} + (\bar{E}^2 - 1) U_T \right]^2 - 1} \right]^{1/2} \quad (6.24)$$

By evaluating (6.24) before performing the numerical integrations, we are able to set the phase lag between  $\bar{\psi}$  and  $\psi_T$  appropriately. This permits us to avoid computing changes which merely reflect a readjustment between the two fields.

Finally, use was made of the frequency equation (6.13) to compute the propagation velocity for the long, stable waves. It was desired to avoid unrealistic retrogression of the long waves in the computational scheme without introducing a mean divergence field. To achieve this the zonal dimension of the integration region was set at 10,000 km. This value was chosen on the basis of the phase speeds for the mean and shear streamfunctions computed from (6.13) for  $U = 25 \text{ m sec}^{-1}$  and  $U_T = 20 \text{ m sec}^{-1}$  (see Table 1).

TABLE 1. The phase velocities of the mean streamfunction,  $\bar{C}$ , and the shear streamfunction,  $C_T$ , as a function of wave length,  $L$ . Negative values indicate motion to the west.

$L [10^3 \text{ km}]$	$\bar{C} [\text{m sec}^{-1}]$	$C_T [\text{m sec}^{-1}]$
10	- 1.66	+ 5.30
11	- 9.36	+ 4.47
12	-17.79	+ 4.28
13	-26.96	+ 4.09
14	-36.86	+ 3.99
15	-47.49	+ 3.90

## 7. Proposed numerical experiments

Since the energetic consistency of the model may be tested without the inclusion of the diabatic heat terms, it is proposed to carry out comparative predictions with and without the energy conversion term  $(\nabla\chi_T \cdot \nabla\psi_T)$  in the omega equation. For these tests, we will vary the initial stream function using wave numbers two and three, separately and in combination.

Following these adiabatic computations, we will run cases with diabatic heating, using both the energetically consistent and inconsistent versions of the base model.

Finally, we will perform a set of experiments using the energetically consistent version of the model with varying fields of diabatic heating. These last experiments are designed to investigate the possibility that a coupling of the diabatic heating to the adiabatic process of baroclinic development can augment the latter in a significant manner.

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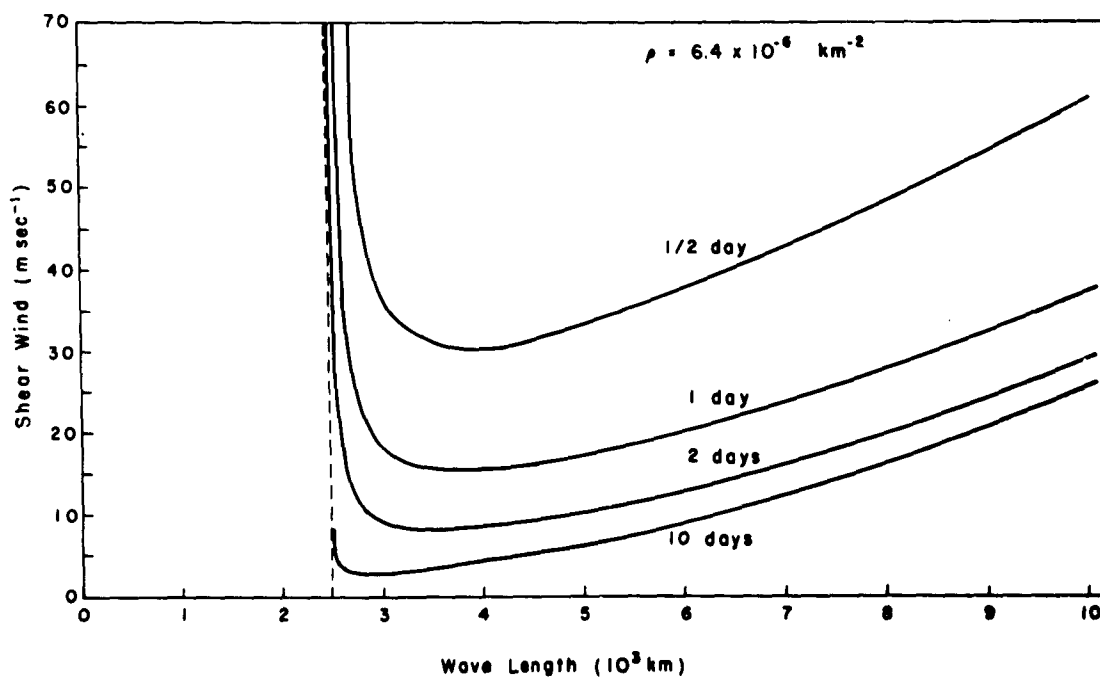
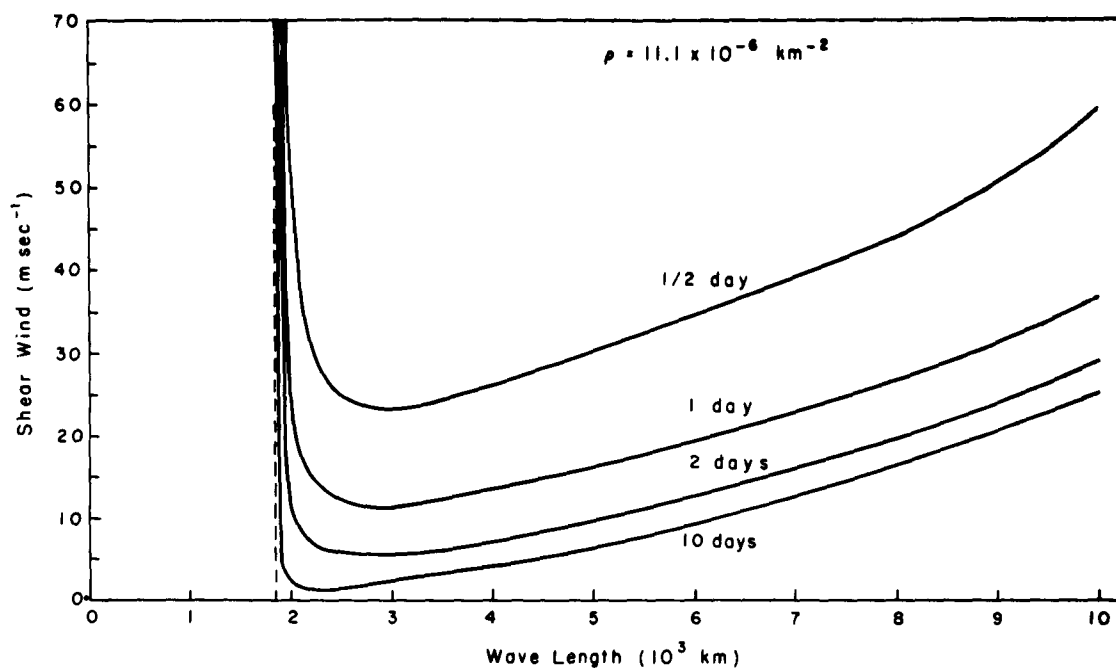


Fig. 1. Stability characteristics of the model showing the wind shear required for a perturbation of any wave length to grow to  $e$  (2.718...) times its initial value in 1/2, 1, 2, and 10 days. Figure A is computed for the parameter,  $\rho$ , equal to its value in the "dry model", while figure B is for the  $\rho$  equal to its value in the "wet model" when condensation is occurring.

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